The Mathematical Mechanic

Sushovan "Sush" Majhi

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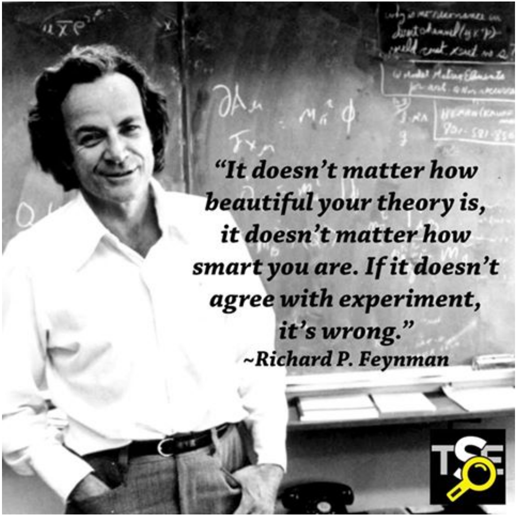
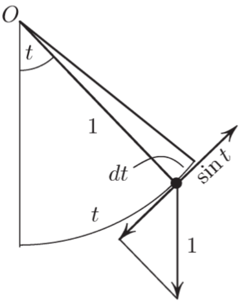
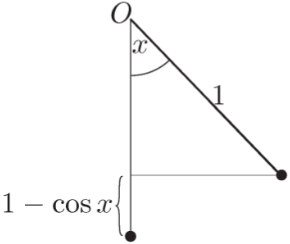

"It doesn't matter how beautiful your theory is, it doesn't matter how smart you are. If it doesn't agree with experiment, it's wrong."
~Richard P. Feynman
TSE

Overview

Consider the two-dimensional simple pendulum,



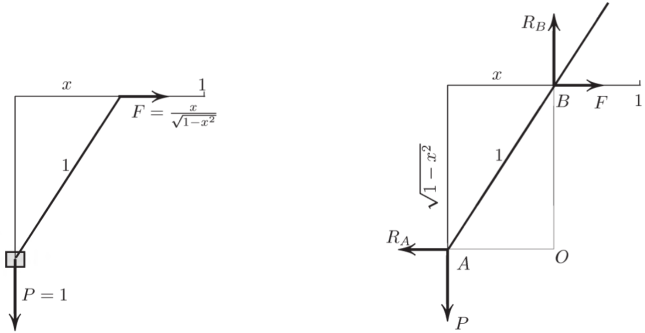
(a) Forces



(b) Torque

Figure : 2

Look at the mechanical structure below



(a) Forces

(b) Torque

Figure : 1

$H : \mathbb{C} \to \mathbb{C}$ can be visualized as a steady two dimensional vector field $X : \mathbb{R}^2 \to \mathbb{R}^2$.

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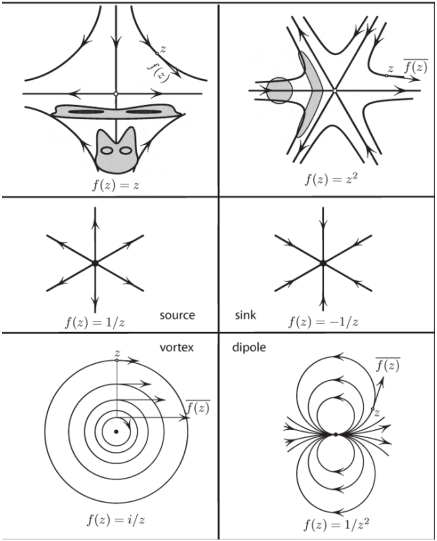
Polya Vector Field

For any complex valued function H of a single complex variable, the vector field associated with $\bar{H}$ is called the Polya vector field of H.

Polya Flows Of Complex Functions



Two Derivatives

Divergence

$$\nabla \cdot X := \frac{\partial X_1}{\partial x} + \frac{\partial X_2}{\partial y}$$

Note: This quantity is invariant under coordinate transformation

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Curl

$$\nabla \times X := \frac{\partial X_2}{\partial x} - \frac{\partial X_1}{\partial y}$$

Note: This quantity is invariant under coordinate transformation

Work Done, Ciculation

For any smooth curve K,

$$\mathcal{W}[X, K] := \int_K X \cdot T ds$$

This is a path/line integral.
When K is a closed curve, this has a special name, Circulation of $\bar{X}$ along K.

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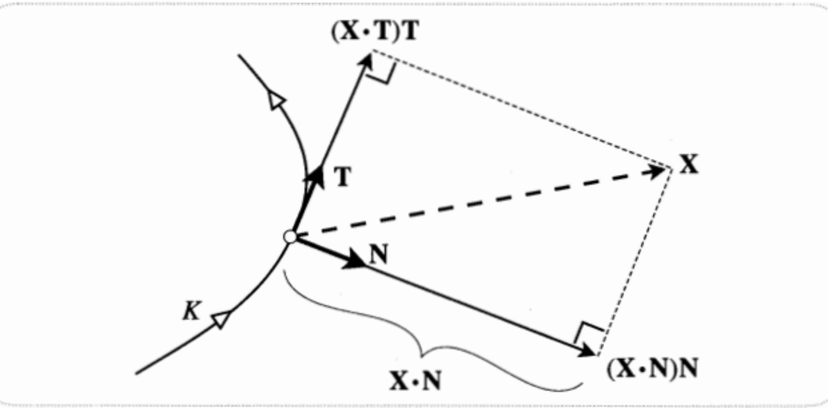
Flux

$$\mathcal{F}[X, K] := \int_K X \cdot N ds$$

It is a surface integral, in general. In our two-dimensional case it it a line integral.


(X•T)T
T
X
N
K
X•N
(X•N)N

How These Derivatives And Integrals Are Connected?

Gauss Divergence Theorem

$$\mathcal{F}[X, K] = \int\int_R [\nabla \cdot X] dA$$

where R is the region encapsulated by the closed curve K.

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Stoke's Theorem(Green's Theorem In Our Case)

$$\mathcal{W}[X, K] = \int\int_R [\nabla \times X] dA$$

Ideal Fluid

A fluid flow is called an Ideal Flow if it is Curl-Free/Irrotational($\nabla \times X = 0$) and Divergence-Free($\nabla \cdot X = 0$) at every point in the region where it is defined.
As a consequence, for an ideal flow X, circulation and flux vanish around any closed curve in the domain.

Ideal Fluid Flow And Analytic Functions

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CR-equations Vs Ideal Flow

CR-equations confirm that The Polya vector field of an analytic function is Ideal!

Complex Line Integral Vs Polya Flow

Let $H : \mathbb{C} \to \mathbb{C}$ be a continuous complex function. Then,

$$\oint_K H(z) = \mathcal{W}[\bar{H}, K] + i\mathcal{F}[\bar{H}, K]$$

where $\bar{H}$ is the Polya vector field of H.

Cauchy's Theorem Vs Ideal Fluid

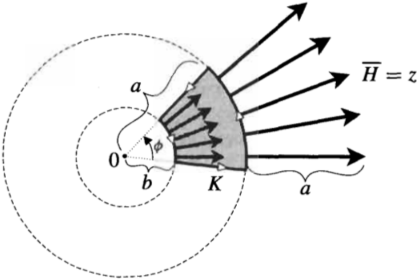
Trivially true.

An example

Let's calculate

$$\oint_K \bar{z} dz = 2iArea(R)$$

where R is the region bounded by K.

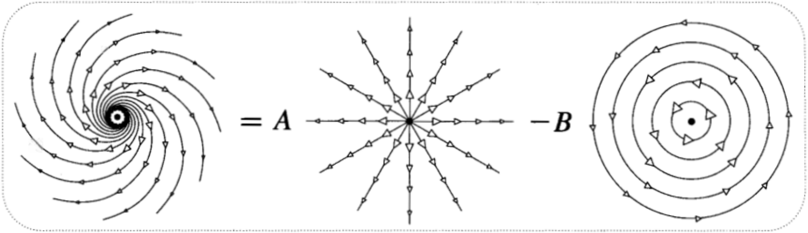


Theorem

Let $H : \mathbb{C} \rightarrow : \mathbb{C}$ be an analytic function then

$$H(w) = \oint_K \frac{H(z)}{z - w} dz$$

where K is a simple closed curve enclosing w in it's interior.



Euler's Sum Via Fluid Flow

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$$\sum_{1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

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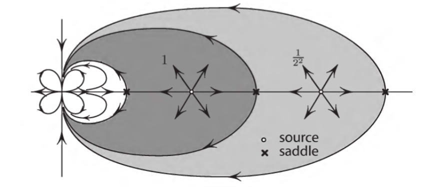
Consider the Polya vector field of

$$H(z) = \frac{\cot \pi z}{2z^2}$$



References

a. "The Mathematical Mechanic", Mark Levi
b. "Visual Complex Analysis", Needham
c. "Lectures On Physics, R. Feynman", Vol 1,2

Thank You!